4301. Let $u = (\ln x)^2$ and $\frac{dv}{dx} = x^{-2}$, so $\frac{du}{dx} = 2\ln x \cdot x^{-1}$ and $v = -x^{-1}$. The parts formula gives

$$I = -\frac{(\ln x)^2}{x} + 2\int \frac{\ln x}{x^2}\,dx.$$

Again by parts, let $u = \ln x$ and $\frac{dv}{dx} = x^{-2}$, so $\frac{du}{dx} = x^{-1}$ and $v = -x^{-1}$. The parts formula gives

$$\begin{aligned} I &= -\frac{(\ln x)^2}{x} - \frac{2\ln}{x} + 2\int x^{-2}\,dx \\ &= -\frac{(\ln x)^2 + 2\ln x + 2}{x} + c. \end{aligned}$$

4302. The generic equation of the trajectory is

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}.$$

The second Pythagorean trig identity gives

$$y = x\tan\theta - \frac{gx^2}{2u^2}(1+\tan^2\theta).$$

Consider this as a quadratic in $\tan\theta$:

$$\begin{aligned} &\frac{gx^2}{2u^2}\tan^2\theta - x\tan\theta + y + \frac{gx^2}{2u^2} = 0 \\ \Longrightarrow\; & gx^2\tan^2\theta - 2u^2x\tan\theta + 2u^2y + gx^2 = 0. \end{aligned}$$

If there is exactly one angle of projection which will reach point (x, y), then the discriminant of the above is zero.

$$\begin{aligned} &\left(-2u^2x\right)^2 - 4\left(gx^2\right)\left(2u^2y + gx^2\right) = 0 \\ \Longrightarrow\; & u^4x^2 - 2gu^2x^2y - g^2x^4 = 0. \end{aligned}$$

For the trajectory, $x \neq 0$, so

$$\begin{aligned} & u^4 - 2gu^2y - g^2x^2 = 0 \\ \Longrightarrow\; y &= \frac{u^4 - g^2x^2}{2gu^2} \\ &\equiv \frac{u^2}{2g} - \frac{gx^2}{2u^2}, \text{ as required.} \end{aligned}$$

— Nota Bene —

The equation of the trajectory quoted at the start is generated by eliminating t from horizontal and vertical *suvat*s. At this level, it's a useful result to know.

4303. A function is concave if its second derivative is −ve. Let f be a polynomial of odd degree $2k+1$, where $k \in \mathbb{N}$. (If $k = 0$, then f is linear, and thus not concave anywhere.) Then f″ is a polynomial of odd degree $2k-1$. The equation $f''(x) = 0$ must, therefore, have at least one real root $x = \alpha$. At $x = \alpha$, the second derivative is not negative, so the function f is not concave. □

4304. The sum of a set of n independent normal variables is normal, with the variance equal to the sum of individual variances:

$$\sum_{i=1}^{n} X_i \sim N(0, n).$$

So, as $n \to \infty$, the variance and therefore standard deviation tends to infinity. Consider the formula

$$z = \frac{x-\mu}{\sigma}.$$

In this question, we have

$$z = \frac{k}{\sigma}.$$

Whatever the value of the constant k, the z value tends to zero as $\sigma \to \infty$. Hence, by symmetry,

$$\lim_{n\to\infty} \mathbb{P}\left(k < \sum_{i=1}^{n} X_i\right) = \frac{1}{2}.$$

4305. The sum of the integers from 1 to ab is

$$\tfrac{1}{2}ab(ab+1).$$

From this we must subtract the integers which are divisible by a. There are b of these in the list: $a, 2a, 3a, ..., ab$. The sum of this sequence is a times the sum of the first b integers, which is $\frac{1}{2}ab(b+1)$. This gives

$$\begin{aligned} &\tfrac{1}{2}ab(ab+1) - \tfrac{1}{2}ab(b+1) \\ \equiv\; &\tfrac{1}{2}ab(ab+1-(b+1)) \\ \equiv\; &\tfrac{1}{2}a(a-1)b^2, \text{ as required.} \end{aligned}$$

4306. We don't know that the sum to infinity exists, so we must work with a limit. The LHS is

$$\begin{aligned} &\sum_{i=1}^{\infty}(b_i - b_{i+1}) \\ =\; &\lim_{n\to\infty}\big((b_1-b_2)+(b_2-b_3)+...+(b_n-b_{n+1})\big) \\ \equiv\; &\lim_{n\to\infty}(b_1 - b_{n+1}) \\ \equiv\; & b_1 - \lim_{n\to\infty} b_{n+1}. \end{aligned}$$

We are told that

$$\lim_{n\to\infty} b_n = k.$$

Shifting the index by one, this also implies that

$$\lim_{n\to\infty} b_{n+1} = k.$$

This gives the limit as $b_1 - k$, as required.

4307. (a) Rearranging the equations,

$$\frac{x}{f(t)} = \cos s, \quad \frac{y}{g(t)} = \sin s.$$

Squaring both equations and adding them, the first Pythagorean trig identity gives

$$\left(\frac{x}{f(t)}\right)^2 + \left(\frac{y}{g(t)}\right)^2 = 1.$$

(b) The ellipse is a transformed unit circle. The functions $\mathrm{f}(t)$ and $\mathrm{g}(t)$ represent the length scale factors in the x and y directions. The area scale factor must be 1, so $\mathrm{f}(t)\,\mathrm{g}(t) = 1$.

4308. (a) The function f is well defined if the sum to infinity converges. $\mathrm{f}(x)$ is a geometric series with common ratio $(1-2x)$. Hence, we require $-1 < 1 - 2x < 1$. Solving this, the largest real domain over which f is well defined is $(0, 2)$.

(b) The sum to infinity is

$$\mathrm{f}(x) = \frac{1-2x}{1-(1-2x)} \equiv \frac{1-2x}{2x} \equiv \frac{1}{2x} - 1.$$

For $x \in (0, 2)$, the range of $\frac{1}{2x}$ is $(1/4, \infty)$. So, the range of f is $(-3/4, \infty)$.

4309. Since $y = x^{2p}$ has $x = 0$ as a line of symmetry and $y = x^{2q+1}$ has the origin as a centre of rotational symmetry, any reflective symmetry can only be in $x = 0$, and any rotational symmetry can only be around O. Algebraically, these symmetries are

$$\begin{aligned} &\text{Even} : \mathrm{f}(-x) = \mathrm{f}(x), \\ &\text{Odd}\ \ : \mathrm{f}(-x) = -\,\mathrm{f}(x). \end{aligned}$$

So, we test $-x$. This gives

$$\begin{aligned} &(-x)^{2p} + (-x)^{2q+1} \\ \equiv\ & x^{2p} - x^{2q+1}. \end{aligned}$$

For reflective symmetry, we would need

$$x^{2p} - x^{2q+1} \equiv x^{2p} + x^{2q+1} \iff x^{2q+1} \equiv 0.$$

For rotational symmetry, we would need

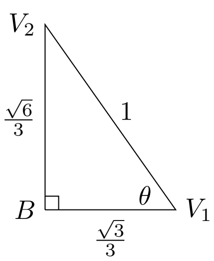
$$x^{2p} - x^{2q+1} \equiv -x^{2p} - x^{2q+1} \iff x^{2p} \equiv 0.$$

Neither of these identities hold. Hence, the graph has neither reflective nor rotational symmetry.

——— Nota Bene ———

The above argument shows why most functions have no symmetry: any mixture of even symmetry ($y = x^{2p}$) and odd symmetry ($y = x^{2q+1}$) results in no symmetry at all.

4310. The centres of the cannonballs are equidistant, so they form a regular tetrahedron. Let this have side length 1. The base is an equilateral triangle, whose medians have length $\sqrt{3}/2$. The centre of the base B, therefore, lies at a distance of $\sqrt{3}/3$ from its vertices:

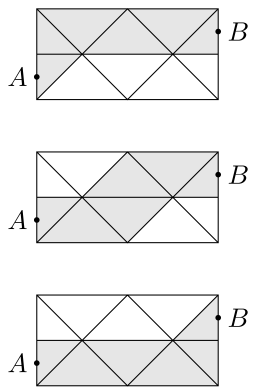


Three reactions of magnitude R act on the top cannonball, each along a side symmetrical to V_1V_2. The horizontal components of these sum to zero by symmetry. The vertical components are identical, and sum to mg. This gives $3R\sin\theta = mg$. And $\sin\theta = \sqrt{6}/3$ from the triangle above, so

$$R = \frac{mg}{\sqrt{6}}, \text{ as required.}$$

4311. The possibility space consists of ${}^{10}\mathrm{C}_6 = 210$ sets of six regions.

There are five regions in the lower half and five in the upper half. Since one border-crossing must go from the lower half to the upper half, the other four border-crossings must be travelling from left to right. There are three border-crossings from the lower half to the upper half, which gives three successful outcomes:



The probability is therefore $\frac{3}{210}$, which is $\frac{1}{70}$.

4312. Combining the integrals,

$$I = \int e^{\tan x}(1 + \tan^2 x)\,dx.$$

The second Pythagorean trig identity gives

$$I = \int e^{\tan x}\sec^2 x\,dx.$$

Noticing that $\sec^2 x$ is the derivative of $\tan x$, we integrate by inspection:

$$I = e^{\tan x} + c.$$

4313. (a) Without loss of generality, take $(1, 0)$ to be the initial position. The final position is then $(\cos 2\theta, \sin 2\theta)$. The distance between these is

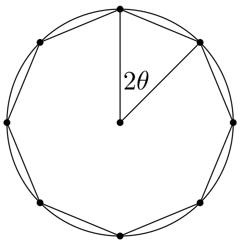
$$\begin{aligned} d &= \sqrt{(\cos 2\theta - 1)^2 + \sin^2\theta} \\ &\equiv \sqrt{\cos^2 2\theta + \sin^2 2\theta - 2\cos 2\theta + 1} \\ &\equiv \sqrt{2 - 2\cos 2\theta}. \end{aligned}$$

(b) Using $\cos 2\theta = 1 - 2\sin^2\theta$, this is

$$\begin{aligned} d &= \sqrt{2 - 2\big(1 - 2\sin^2\theta\big)} \\ &\equiv \sqrt{4\sin^2\theta} \\ &\equiv 2\sin\theta. \end{aligned}$$

(c) A full rotation is 2π radians, so $2n\theta = 2\pi$, which gives $n = \frac{\pi}{\theta}$.

(d) The scenario is as follows, with 2θ subtended by each sector.



From prior results, the polygon has perimeter $2n\sin\theta$. The circumference of the circle is 2π. As $\theta \to 0$, $n \to \infty$. In this limit, the ratio of circumference and perimeter tends to 1:

$$\begin{aligned} &\lim_{\theta\to 0} \frac{2n\sin\theta}{2\pi} = 1 \\ \Longrightarrow\ &\lim_{\theta\to 0} \frac{2\frac{\pi}{\theta}\sin\theta}{2\pi} = 1 \\ \Longrightarrow\ &\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1, \text{ as required.} \end{aligned}$$

4314. Let $u = (\ln x)^2$ and $\frac{dv}{dx} = 1$. So, $v = x$ and

$$\frac{du}{dx} = \frac{2\ln x}{x}.$$

The parts formula gives

$$\begin{aligned} &\int (\ln x)^2\,dx \\ &= x(\ln x)^2 - \int x \cdot \frac{2\ln x}{x}\,dx \\ &= x(\ln x)^2 - 2\int \ln x\,dx. \end{aligned}$$

Here, we quote the result

$$\int \ln x\,dx = x\ln x - x + c.$$

This gives

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2x\ln x + 2x + c.$$

— Nota Bene —

The proof of the quoted result above is again parts, in exactly the same manner as this problem, i.e. setting $u = \ln x$ and $\frac{dv}{dx} = 1$.

4315. Let $z = \sec x$ and $y = \tan x$. The derivatives of these are $\frac{dz}{dx} = \sec x\tan x$ and $\frac{dy}{dx} = \tan^2 x$. So, by the chain rule,

$$\begin{aligned} \frac{dz}{dy} &= \frac{dz}{dx} \div \frac{dy}{dx} \\ &= \frac{\sec x \tan x}{\tan^2 x} \\ &\equiv \frac{1}{\cos x \cdot \frac{\sin x}{\cos x}} \\ &\equiv \frac{1}{\sin x} \\ &\equiv \operatorname{cosec} x, \text{ as required.} \end{aligned}$$

4316. For f to be well defined, the radicand $x^4 + x^2 - 6$ must be +ve. The boundary equation is

$$\begin{aligned} &x^4 + x^2 - 6 = 0 \\ \Longrightarrow\ &(x^2 + 3)(x^2 - 2) = 0 \\ \Longrightarrow\ &x^2 = -3, 2. \end{aligned}$$

The former offers no real roots, so $x = \pm\sqrt{2}$. The radicand is positive iff $|x| > \sqrt{2}$. So, the largest real domain over which it is possible to define the function is $\big(-\infty, -\sqrt{2}\big) \cup \big(\sqrt{2}, \infty\big)$.

4317. The low numbers $\{1, ..., 10\}$ constitute half of the numbers. For none to be adjacent, the low and high numbers must make a chequerboard pattern, alternating low/high. But five faces surround one vertex. Five is odd, so this is not possible. Hence, the probability is zero.

4318. Solving for intersections,

$$\begin{aligned} &-ky = k + y^2 \\ \Longrightarrow\ &y^2 + ky + k = 0 \\ \Longrightarrow\ &y = \frac{-k \pm \sqrt{k^2 - 4k}}{2}. \end{aligned}$$

The x values are then given by $x^2 = -ky$. For exactly two points of intersection, the quadratic in y needs to have at least one root.

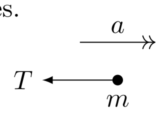
- For one root, $k^2 - 4k = 0$, so $k = 0, 4$. If $k = 0$, then everything is zero, and there is only one intersection, at the origin. If $k = 4$, then $y = -2$. This gives $x^2 = 4$, so two points of intersection at $(\pm 2, -2)$.
- For two roots, $k^2 - 4k > 0$, so $k < 0$ or $k > 4$.
  - If $k < 0$, then $\sqrt{k^2 - 4k} > k$, so one y value is positive and the other negative. This gives two x values.
  - If $k > 4$, then both y values are negative, which gives four x values.

Collating this information, $k \in (-\infty, 0) \cup \{4\}$.

4319. The derivative is $-\sin x$. So, the gradient of the normal at $x = a$ is $1/\sin a$. Since a is close to zero, we can approximate this as $1/a$. The coordinates at $x = a$ are $(a, \cos a)$, which we can approximate by $\left(a, 1 - \frac{1}{2}a^2\right)$. So, the equation of the normal can be approximated by

$$\begin{aligned} & y - \left(1 - \tfrac{1}{2}a^2\right) = \tfrac{1}{a}(x - a) \\ \Longrightarrow\ & ay - a + \tfrac{1}{2}a^3 + a - x = 0 \\ \Longrightarrow\ & 2ay - 2x + a^3 = 0, \text{ as required.} \end{aligned}$$

4320. (a) In 1D, the forces are as follows. The variable a is defined with a +ve value of a corresponding to a +ve second rate of change of extension x. The actual direction of the acceleration will vary, with $\pm$ values of a (and $\mp$ values of T) at different times.



The magnitude of the tension is given, using Hooke's law, by $T = kx$, where k is a positive constant of proportionality. NII is $-kx = ma$, which is $a = -\frac{k}{m}x$. Since both k and m are positive, we can write $a = -qx$, where $q > 0$.

(b) Differentiating the proposed solution,

$$\begin{aligned} & x = A\sin\sqrt{q}t + B\cos\sqrt{q}t \\ \Longrightarrow\ & \dot{x} = A\sqrt{q}\cos\sqrt{q}t - B\sqrt{q}\sin\sqrt{q}t \\ \Longrightarrow\ & \ddot{x} = -Aq\sin\sqrt{q}t - Bq\cos\sqrt{q}t \\ & \equiv -q(A\sin\sqrt{q}t + B\cos\sqrt{q}t) \\ & = -qx. \end{aligned}$$

So, the proposed solution curve satisfies the differential equation.

(c) The solution curve is a sinusoidal oscillation around $x = 0$, so the mass oscillates around the spring's natural length. Resistances have been neglected; hence, the model suggests that this oscillation continues indefinitely.

4321. The gradient at $\left(p, p^3\right)$ is $3p^2$, so the equation of the tangent is $y - p^3 = 3p^2(x - p)$. Solving for intersections with the curve,

$$\begin{aligned} & x^3 - p^3 = 3p^2(x - p) \\ \Longrightarrow\ & x^3 - 3p^2x + 2p^3 = 0. \end{aligned}$$

We know that this has a root at $x = p$, so

$$(x - p)(x^2 + px - 2p^2) = 0.$$

For a re-intersection with the curve, we need the equation $x^2 + px - 2p^2 = 0$ to have at least one root (that isn't $x = p$). So, we require

$$\Delta = p^2 + 8p^2 = 9p^2 \geq 0.$$

This holds for all p. If $p = 0$, however, then the quadratic provides no new intersections beyond $x = p$. So, $p = 0$ is the only value for which the tangent does not re-intersect the curve. □

4322. Using a calculator, $\mathbb{P}(X > 0) = 0.158655....$ By symmetry of the distributions (with each other), this is also $\mathbb{P}(Y < 0)$. Assuming independence, the probability of both $X > 0$ and $Y < 0$ is

$$\begin{aligned} & \mathbb{P}(X > 0) \times \mathbb{P}(Y < 0) \\ = \ & 0.158655^2 \\ = \ & 0.02517... \end{aligned}$$

This is close to the given value of $\frac{1}{40}$, which is 0.025. Since the number of trials run is small, this level of discrepancy is certainly consistent with the assumption of independence.

4323. (a) Consider a small-angle approximation. Since $\sin x \approx x$, the factor $(x - \sin x) \approx 0$. So, the first claim is certainly true. In particular, the approximation is exactly true in the limit as $x \to 0$.

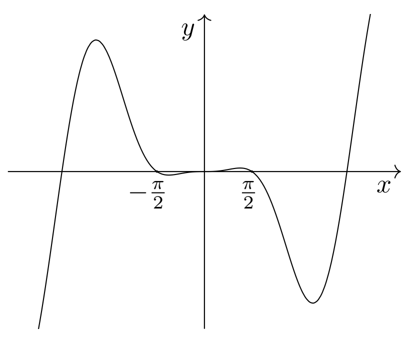
(b) For the second claim, there are x values in the domain which cannot be considered small. So, consider the range of f. It has odd symmetry, so we need only consider the domain $\left[0, \pi/2\right]$. The value of $\mathrm{f}(x)$ is zero at the endpoints of this domain, and there are no discontinuities. So, the range is determined by the stationary value of $\mathrm{f}(x)$ over $(0, \pi/2)$. We are told there is exactly one such value. Looking for SPs,

$$\begin{aligned} & -\sin x(x - \sin x) + \cos x(1 - \cos x) = 0 \\ \Longrightarrow\ & -x\sin x - \cos 2x + \cos x = 0. \end{aligned}$$

This equation is not analytically solvable. The N-R iteration is $x_{n+1} = x_n - \mathrm{F}(x_n)$, where $\mathrm{F}(x_n)$ is

$$-\frac{-x_n\sin x_n - \cos 2x_n + \cos x_n}{-\sin x_n - x_n\cos x_n + 2\sin 2x_n + \sin x_n}.$$

Running this with $x_0 = 1.5$, we get $x_1 = 1.26...$ and then $x_n \to 1.17964$. So, the stationary value of $\mathrm{f}(x)$ is $\mathrm{f}(1.17964) = 0.09728....$ Hence, to 3sf, the range of f over the domain $[\pi/2, \pi/2]$ is $[-0.0973, 0.0973]$. So, for some purposes, it may be reasonable to assume that $\mathrm{f}(x) \approx 0$ over this domain.



4324. Substituting for x, the equation for intersections is

$$\begin{aligned}&(y^2+y)+y=\big((y^2+y)-y\big)^2+k\\ \Longrightarrow\ &y^2+2y=y^4+k\\ \Longrightarrow\ &y^4-y^2-2y+k=0.\end{aligned}$$

Consider $\mathrm{f}(y)=y^4-y^2-2y+k$. We need this function to have a repeated root. At such roots, the derivative is zero. Looking for SPs,

$$\begin{aligned}&4y^3-2y-2=0\\ \Longrightarrow\ &y=1.\end{aligned}$$

The point of tangency must therefore be at $y=1$. Substituting this into both equations gives $(1,1)$. Hence, $k=2$.

4325. Separating the variables,

$$\begin{aligned}&2e^{-t}\frac{dP}{dt}-\frac{t}{P}=0\\ \Longrightarrow\ &2e^{-t}\frac{dP}{dt}=\frac{t}{P}\\ \Longrightarrow\ &2P\frac{dP}{dt}=te^t\\ \Longrightarrow\ &P^2=\int te^t\,dt.\end{aligned}$$

Integrating by parts,

$$P^2=e^t(t-1)+c.$$

Substituting $t=0$, $P=100$, we get $c=10001$. So, the value of P predicted at $t=10$ is given by

$$\begin{aligned}P^2&=e^{10}(10-1)+10001\\ &\approx 208239.\end{aligned}$$

To the nearest whole number, $P=456$.

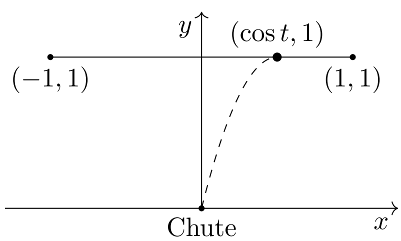
4326. ① $\Longleftrightarrow$ ②. The factor theorem tells us that the following statements are equivalent:

① $\mathrm{f}(x)-\mathrm{g}(x)=0$ has a root at $x=a$,

② $\mathrm{f}(x)-\mathrm{g}(x)$ has a factor of $(x-a)$.

The former statement is the same as $\mathrm{f}(a)=\mathrm{g}(a)$.

4327. The scenario, with a trajectory after release shown as a dashed line, is



The velocity of the prize, while held in the jaws, is $\dot{x}=-\sin t$. So, upon release at time t_0, the position is $(\cos t_0,1)$ and the velocity is $-\sin t_0$ horizontally. So, at time $t\geq t_0$,

$$\begin{aligned}x&=\cos t_0-t\sin t_0,\\ y&=1-\tfrac{1}{2}gt^2.\end{aligned}$$

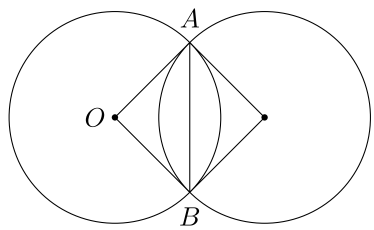
We need the trajectory to pass through the origin. Substituting $(0,0)$ gives

$$\begin{aligned}0&=\cos t_0-t\sin t_0,\\ 0&=1-\tfrac{1}{2}gt^2.\end{aligned}$$

The first equation gives the time of landing at the mouth of the chute as $t=\cot t_0$. Substituting this into vertical equation,

$$\begin{aligned}&0=1-\tfrac{1}{2}g\cot^2 t_0\\ \Longrightarrow\ &\tan^2 t_0=\frac{g}{2}\\ \Longrightarrow\ &\tan t_0=\pm\sqrt{\frac{g}{2}},\text{ as required.}\end{aligned}$$

4328. The area of each unshaded intersection is the same. Each can be seen as two segments subtending a right angle at the centre.



Sector OAB has area $\frac{1}{4}\pi r^2$. $\triangle OAB$ has area $\frac{1}{2}r^2$. So, minor segment AB has area $\frac{1}{4}\pi r^2-\frac{1}{2}r^2$. The total intersection, therefore, has area

$$\begin{aligned}A_{\text{int}}&=2(\tfrac{1}{4}\pi r^2-\tfrac{1}{2}r^2)\\ &=\big(\tfrac{\pi}{2}-1\big)r^2.\end{aligned}$$

The pattern consists of nine circles, each of area πr^2. From this we subtract twelve intersections, which have been overcounted. So, the total area is

$$\begin{aligned}A_{\text{pattern}}&=9\pi r^2-12A_{\text{int}}\\ &=9\pi r^2-12\big(\tfrac{\pi}{2}-1\big)r^2\\ &=3r^2(\pi+4),\text{ as required.}\end{aligned}$$

4329. (a) Differentiating the definition,

$$\begin{aligned}&\sinh x=\frac{e^x-e^{-x}}{2}\\ \Longrightarrow\ &\cosh x=\frac{e^x+e^{-x}}{2}.\end{aligned}$$

(b) For SPs of the sinh graph,

$$\frac{e^x+e^{-x}}{2}=0.$$

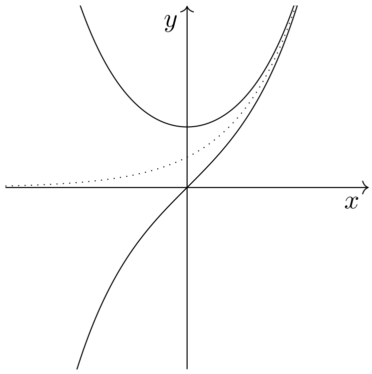
Since e^x and e^{-x} are both positive, this has no roots. So, hyperbolic sine is never stationary.

For SPs of the cosh function,

$$\begin{aligned} \frac{e^x - e^{-x}}{2} &= 0 \\ \Longrightarrow e^x &= e^{-x} \\ \Longrightarrow e^{2x} &= 1 \\ \Longrightarrow x &= 0. \end{aligned}$$

So, $y = \cosh x$ has one SP at $(0, 1)$.

(c) The graph $y = \sinh x$ has odd symmetry. It passes through O and is always increasing. The graph $y = \cosh x$ has even symmetry, with a global minimum at $x = 1$. As $x \to \infty$, both curves tend towards $y = \frac{1}{2}e^x$.



4330. (a) For $n = 2$, $f_2(x) = kx(1-x)^2$. Multiplying this out, $f_2(x) = k(x - 2x^2 + x^3)$. So, we require

$$\begin{aligned} k\int_0^1 x - 2x^2 + x^3\,dx &= 1 \\ \Longrightarrow k\left[\tfrac{1}{2}x^2 - \tfrac{2}{3}x^3 + \tfrac{1}{4}x^4\right]_0^1 &= 1 \\ \Longrightarrow \tfrac{1}{12}k &= 1 \\ \Longrightarrow k &= 12. \end{aligned}$$

(b) The same calculation, now in general, is

$$k\int_0^1 x(1-x)^n\,dx = 1.$$

To find a general formula, we integrate by parts. Let $u = x$ and $\frac{dv}{dx} = (1-x)^n$, so that $\frac{du}{dx} = 1$ and $v = -\frac{1}{n+1}(1-x)^{n+1}$. The parts formula gives

$$\begin{aligned} &\int_0^1 x(1-x)^n\,dx \\ &\equiv \left[\tfrac{-1}{n+1}x(1-x)^{n+1}\right]_0^1 + \int_0^1 \tfrac{1}{n+1}(1-x)^{n+1}\,dx \\ &\equiv \left[\tfrac{-1}{n+1}x(1-x)^{n+1} - \tfrac{1}{(n+1)(n+2)}(1-x)^{n+2}\right]_0^1 \\ &\equiv (0-0) - \left(0 - \tfrac{1}{(n+1)(n+2)}\right). \end{aligned}$$

So, $k = (n+1)(n+2)$.

4331. Using the conditional probability formula,

$$\mathbb{P}(X \le 4 \mid X > 2) = \frac{\mathbb{P}(2 < X \le 4)}{\mathbb{P}(X > 2)}.$$

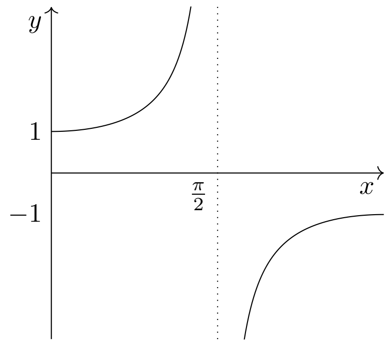
The probability distribution of $B(5, 1/3)$ is

x	0	1	2	3	4	5
$\mathbb{P}(X = x)$	$\frac{32}{243}$	$\frac{80}{243}$	$\frac{80}{243}$	$\frac{40}{243}$	$\frac{10}{243}$	$\frac{1}{243}$

This gives

$$\mathbb{P}(X \le 4 \mid X > 2) = \frac{\frac{40}{243} + \frac{10}{243}}{\frac{40}{243} + \frac{10}{243} + \frac{1}{243}} = \frac{50}{51}.$$

4332. Secant is not invertible over $\mathbb{R}$. Restricted to the appropriate domain and codomain, the graph of its invertible version is



So, arcsec has domain $(-\infty, -1] \cup [1, \infty)$ and range $[0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$.

To find the domain of f, set $2x + 1 = \pm 1$. This gives boundaries at $x = 0, -1$. Hence, the largest real domain over which $f(x) = \text{arcsec}(2x+1)$ may be defined is $(-\infty, -1] \cup [0, \infty)$.

A linear input transformation such as $x \mapsto 2x + 1$ leaves the range unchanged: it is $[0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$.

4333. (a) The horizontal and vertical equations are

$$\begin{aligned} 2 &= (8\cos\theta)t, \\ 1 &= (8\sin\theta)t - \tfrac{1}{2}gt^2. \end{aligned}$$

Substituting the former into the latter,

$$\begin{aligned} 1 &= \frac{8\sin\theta}{4\cos\theta} - \frac{g}{2 \cdot 4^2\cos^2\theta} \\ \Longrightarrow 1 &= 2\tan\theta - \tfrac{g}{32}\sec^2\theta \\ \Longrightarrow 1 &= 2\tan\theta - \tfrac{g}{32}(1 + \tan^2\theta) \\ \Longrightarrow 320 &= 640\tan\theta - 98 - 98\tan^2\theta \\ \Longrightarrow 49\tan^2\theta &- 320\tan\theta + 209 = 0. \end{aligned}$$

(b) Using the quadratic formula,

$$\begin{aligned} \tan\theta &= 0.73609, 5.7945, \\ \therefore\quad \theta &= 36.4°, 80.2° \text{ (1dp)}. \end{aligned}$$

(c) The equation of the trajectory is

$$y = (\tan\theta)x - \tfrac{g}{64}(1 + \tan^2\theta)x^2.$$

Its gradient is

$$\frac{dy}{dx} = \tan\theta - \tfrac{g}{32}(1 + \tan^2\theta)x.$$

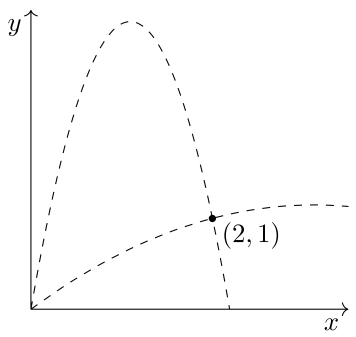
At $\tan\theta = 0.73609$, this is

$$\begin{aligned}\frac{dy}{dx} &= 0.73609 - \tfrac{g}{32}(1 + 0.73609^2)\\ &= 0.264 > 0.\end{aligned}$$

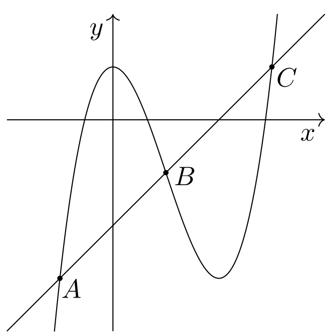
Since the gradient is positive, the basketball is still travelling upwards when it reaches the hoop. On the other hand, at $\tan\theta = 5.7945$, the gradient is

$$\begin{aligned}\frac{dy}{dx} &= 5.7945 - \tfrac{g}{32}(1 + 5.7945^2)\\ &= -10.60 < 0.\end{aligned}$$

Since the gradient is (significantly) less than zero, the basketball can drop down through the hoop.



4334. Any cubic graph has rotational symmetry around its point of inflection, which is B. A straight line through B also has rotational symmetry, which means that the problem has rotational symmetry about B. Therefore, both the x coordinates and y coordinates of A, B, C must be in AP.



4335. We integrate by parts, by the tabular integration method. Let $u = x^2 + 1$ and $v'' = \cos 4x$. Then the derivatives and integrals are given by

Signs	Derivatives	Integrals
$+$	$x^2 + 1$	$\cos 4x$
$-$	$2x$	$\frac{1}{4}\sin 4x$
$+$	2	$-\frac{1}{16}\cos 4x$
$-$	0	$-\frac{1}{64}\sin 4x.$

This gives

$$\begin{aligned}&\int_0^\pi (x^2 + 1)\cos 4x\,dx\\ &= \left[\tfrac{1}{4}(x^2+1)\sin 4x + \tfrac{1}{8}x\cos 4x - \tfrac{1}{32}\sin 4x\right]_0^\pi\\ &= (0 + \tfrac{\pi}{8} - 0) - (0)\\ &= \tfrac{\pi}{8}, \text{ as required.}\end{aligned}$$

— Nota Bene —

The tabular integration method is significantly faster than brute force when performing multiple integrations by parts. Derivatives and integrals are calculated until the derivative column reads zero. Terms are then read off as products. Firstly,

Signs	Derivatives	Integrals
$+$	$x^2 + 1$	$\cos 4x$
$-$	$2x$	$\frac{1}{4}\sin 4x$

Secondly,

Signs	Derivatives	Integrals
$+$	$x^2 + 1$	$\cos 4x$
$-$	$2x$	$\frac{1}{4}\sin 4x$
$+$	2	$-\frac{1}{16}\cos 4x$

The procedure continues until all subsequent terms are zero.

4336. Differentiating implicitly,

$$2x + 2 + 3y^2\frac{dy}{dx} = y + x\frac{dy}{dx}.$$

Setting $\frac{dy}{dx} = 0$ for SPs,

$$2x + 2 = y.$$

Substituting this into the equation of the curve,

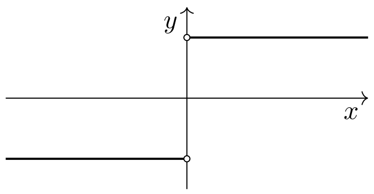
$$\begin{aligned}&x^2 + 2x + (2x+2)^3 = 63 + x(2x+2)\\ \Longrightarrow\ &8x^3 + 23x^2 + 24x - 55 = 0\\ \Longrightarrow\ &(x-1)(8x^2 + 31x + 55) = 0.\end{aligned}$$

The quadratic has $\Delta = -799 < 0$, so has no real roots. The only SP is at $x = 1$. Subbing back in,

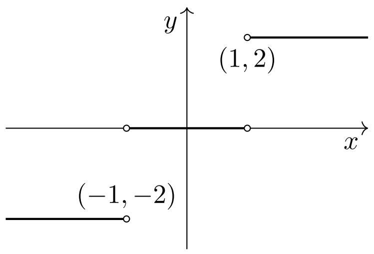
$$\begin{aligned}&3 + y^3 = 63 + y\\ \Longrightarrow\ &y^3 - y - 60 = 0\\ \Longrightarrow\ &(y-4)(y^2 + 4y + 15) = 0.\end{aligned}$$

This quadratic has $\Delta = -44 < 0$, also has no real roots. Hence, the only SP is at $(1, 4)$.

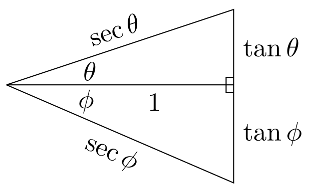
4337. The graph of the function $f(x) = \frac{|x|}{x}$ is



The two terms of the required graph are translated versions of this step function, with steps at $x = -1$ and $x = 1$ respectively. This gives a "two-step" function:



4338. Two of the sides are $\sec\theta$ and $\sec\phi$, and the third is $(\tan\theta + \tan\phi)$.



Using the two right-angled triangles,

$$\begin{aligned} c^2 &= (\tan\theta + \tan\phi)^2 \\ &\equiv \tan^2\theta + 2\tan\theta\tan\phi + \tan^2\phi. \end{aligned}$$

Using the cosine rule,

$$\cos(\theta+\phi) = \frac{\sec^2\theta + \sec^2\phi - c^2}{2\sec\theta\sec\phi}.$$

The numerator is

$$\begin{aligned} &\sec^2\theta + \sec^2\phi - \tan^2\theta - 2\tan\theta\tan\phi - \tan^2\phi \\ \equiv\ &\sec^2\theta - \tan^2\theta + \sec^2\phi - \tan^2\phi - 2\tan\theta\tan\phi \\ \equiv\ &2 - 2\tan\theta\tan\phi. \end{aligned}$$

This gives

$$\begin{aligned} \cos(\theta+\phi) &\equiv \frac{2 - 2\tan\theta\tan\phi}{2\sec\theta\sec\phi} \\ &\equiv \cos\theta\cos\phi - \sin\theta\sin\phi. \end{aligned}$$

4339. The possibility space consists of $2^6 = 64$ outcomes. The successful outcomes, with a run of heads, are as follows, classified by length of run:

4	5	6
HHHHTH	HHHHHT	HHHHHH
HHHHTT	THHHHH	
THHHHT		
TTHHHH		
HTHHHH		

These are repeated symmetrically with runs of tails. So, the probability is $\frac{16}{64} = \frac{1}{4}$.

4340. For the particle to come to rest, both components of velocity must be zero.

$$\begin{aligned} (2k+1)\cos(2k+1)t &= 0, \\ -(2k+3)\sin(2k+3)t &= 0. \end{aligned}$$

Simplifying these,

$$\begin{aligned} \cos(2k+1)t &= 0, \\ \sin(2k+3)t &= 0. \end{aligned}$$

So, for $m, n \in \mathbb{Z}$:

$$\begin{aligned} (2k+1)t &= \tfrac{\pi}{2} + m\pi, \\ (2k+3)t &= n\pi. \end{aligned}$$

These give

$$\begin{aligned} t &= \left(\frac{\frac{1}{2} + m}{2k+1}\right)\pi, \\ t &= \left(\frac{n}{2k+3}\right)\pi. \end{aligned}$$

Hence, both components are zero if and only if, for some integers m and n,

$$\begin{aligned} &\frac{\frac{1}{2} + m}{2k+1} = \frac{n}{2k+3}, \\ \iff &(1+2m)(2k+3) = 2n(2k+1). \end{aligned}$$

But the LHS is odd, while the RHS is even. So, there are no integers m and n which satisfy the above equation. Hence, there are no t values for which the particle is at rest.

4341. Multiplying up, the equation is

$$\begin{aligned} &(x^4 + 3x^2 + 2)(x^2 - 3x + 2) \\ &\qquad = (x^4 - 3x^2 + 2)(x^2 + 3x + 2). \end{aligned}$$

The LHS is

$$x^6 - 3x^5 + 5x^4 - 9x^3 + 8x^2 - 6x + 4.$$

The RHS is

$$x^6 + 3x^5 - x^4 - 9x^3 - 4x^2 + 6x + 4.$$

Subtracting the LHS from the RHS, the terms in x^6, x^3 and x^0 cancel, leaving

$$\begin{aligned} &6x^5 - 6x^4 - 12x^2 + 12x = 0 \\ \implies &x(x^4 - x^3 - 2x + 2) = 0. \end{aligned}$$

Using a polynomial solver on the quartic factor, the solution is $x = 0, 1, \sqrt[3]{2}$. However, $x = 1$ does not satisfy the original equation, as it is a root of $x^2 - 3x + 2$, for which the original equation is undefined.

So, the solution is $x \in \{0, \sqrt[3]{2}\}$.

4342. Both techniques work. The solutions are:

① By parts, let $u = x$ and $v' = \sqrt{2x-3}$, so that $u' = 1$ and $v = \frac{1}{3}(2x-3)^{\frac{3}{2}}$. The parts formula gives

$$\begin{aligned}&\int x\sqrt{2x-3}\,dx\\ &= \tfrac{1}{3}x(2x-3)^{\frac{3}{2}} - \tfrac{1}{3}\int (2x-3)^{\frac{3}{2}}\,dx\\ &= \tfrac{1}{3}x(2x-3)^{\frac{3}{2}} - \tfrac{1}{15}(2x-3)^{\frac{5}{2}} + c\\ &\equiv \tfrac{1}{5}(x+1)(2x-3)^{\frac{3}{2}} + c.\end{aligned}$$

② By substitution, let $u = 2x-3$. So, $\frac{1}{2}\,du = dx$ and $x = \frac{1}{2}(3+u)$. Enacting the substitution,

$$\begin{aligned}&\int x\sqrt{2x-3}\,dx\\ &= \int \tfrac{1}{2}(3+u)\sqrt{u}\cdot\tfrac{1}{2}\,du\\ &= \tfrac{1}{4}\int 3u^{\frac{1}{2}} + u^{\frac{3}{2}}\,du\\ &= \tfrac{1}{4}\left(2u^{\frac{3}{2}} + \tfrac{2}{5}u^{\frac{5}{2}}\right) + c\\ &= \tfrac{1}{2}(2x-3)^{\frac{3}{2}} + \tfrac{1}{10}(2x-3)^{\frac{5}{2}} + c\\ &\equiv \tfrac{1}{5}(x+1)(2x-3)^{\frac{3}{2}} + c.\end{aligned}$$

4343. The curve is symmetrical, so the point of tangency in the positive quadrant lies on the line $y = x$. Solving this simultaneously with the curve,

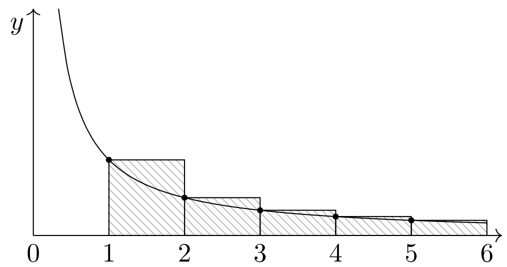
$$\begin{aligned}&x^{\frac{1}{2n}} + x^{\frac{1}{2n}} = 1\\ \Longrightarrow\ &x^{\frac{1}{2n}} = \tfrac{1}{2}\\ \Longrightarrow\ &x = \left(\tfrac{1}{2}\right)^{2n}.\end{aligned}$$

So, the radii of the circles are $r_n = \sqrt{2}\left(\frac{1}{2}\right)^{2n}$. This gives the areas of the circles as

$$A_n = 2\left(\tfrac{1}{2}\right)^{4n}\pi.$$

These areas A_n are all positive, in decreasing GP. Hence, as $n \to \infty$, $A_n \to 0$. □

4344. (a) The sum can be represented as a rectangular approximation to the integral given, with the upper-left corners of the rectangles placed on the curve $y = \frac{1}{x}$:



The sum of the areas of the first n rectangles is the partial sum S_n. This sum is greater than the corresponding definite integral. Since the right-hand edge of the nth rectangle is at $x = n+1$, this gives

$$S_n > \int_1^{n+1} \frac{1}{x}\,dx.$$

(b) The integral on the RHS is

$$\begin{aligned}&\int_1^{n+1} \frac{1}{x}\,dx\\ \equiv\ &\Big[\ln|x|\Big]_1^{n+1}\\ \equiv\ &\ln|n+1| - \ln|1|\\ \equiv\ &\ln(n+1).\end{aligned}$$

The range of the logarithm function is $\mathbb{R}$. So, as $n \to \infty$, the value of the integral tends to infinity. Since the partial sum S_n is greater than the integral, this also tends to infinity, i.e. it diverges.

4345. Multiplying out, we use $|x|^2 \equiv x^2$:

$$\begin{aligned}&\left(|\sin x| + \tfrac{1}{2}\right)\left(|\sin x| - \tfrac{1}{2}\right) = 0\\ \Longrightarrow\ &|\sin x|^2 - \tfrac{1}{4} = 0\\ \Longrightarrow\ &\sin^2 x = \tfrac{1}{4}\\ \Longrightarrow\ &\sin x = \pm\tfrac{1}{2}\\ \therefore\ \ &x = \tfrac{\pi}{6}, \tfrac{5\pi}{6}, \tfrac{7\pi}{6}, \tfrac{11\pi}{6}.\end{aligned}$$

4346. (a) This is false. Consider $\{0,0,0,0\}$ and $\{1,2,3\}$. The combined median is $m_c = 0$, so $m_1 \not< m_c$.

(b) This is false. Consider $\{0,0,0,0\}$ and $\{0,1\}$. The combined IQR is $I_c = 0$, so $I_1 \not< I_c$.

4347. (a) Taking logs of the proposed relationship,

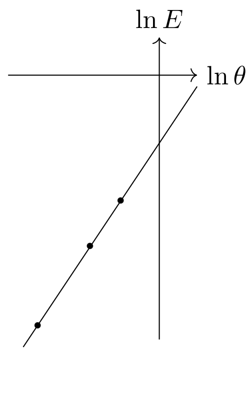
$$\begin{aligned}&E \approx k\theta^3\\ \Longrightarrow\ &\ln E = \ln(k\theta^3)\\ & = \ln k + 3\ln\theta.\end{aligned}$$

This is a linear relationship between $\ln E$ and $\ln\theta$.

(b) The values of $\ln E$ and $\ln\theta$ are

θ	0.2	0.4	0.6
$\ln\theta$	−1.609	−0.916	−0.511
E	0.0013	0.0106	0.0354
$\ln E$	−6.645	−4.547	−3.341

Plotting these with a line of best fit:



The line of best fit passes exactly through the points. So, using the two outermost points, the gradient is

$$m = \frac{-3.341 - (-6.645)}{-0.511 - (-1.609)} = 3.009.$$

So, the equation of the line is

$$\begin{aligned} & \ln E + 6.645 = 3.009(\ln\theta + 1.609) \\ \Longrightarrow\ & \ln E = 3.009 \ln\theta - 1.804. \end{aligned}$$

Comparing this to $\ln E = \ln k + 3\ln\theta$, we can verify that, to a very good approximation, a cubic relationship holds: the line of best fit is of this form, with 3.009 very close to 3.

The coefficient is given by $\ln k = -1.806$, so $k = e^{-1.806} = 0.164 \approx \frac{1}{6}$, as required.

— Nota Bene —

This corresponds to the fact that, increasing the accuracy beyond the small-angle approximation $\sin\theta \approx \theta$, the next approximation is cubic: it is

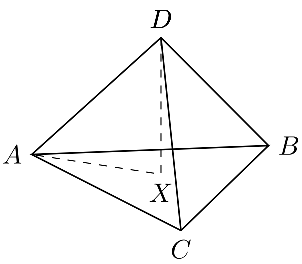
$$\sin\theta \approx \theta - \tfrac{1}{6}\theta^3.$$

4348. We can write the given information in an equation:

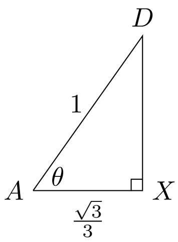
$$\mathrm{p}(x) = (x - \alpha)\,\mathrm{q}(x) + r,$$

where $\mathrm{q}(x)$ is a polynomial in x and r is a constant remainder. Substituting $x = \alpha$ into this equation gives $\mathrm{p}(\alpha) = r$, which is the required result.

4349. Dropping a perpendicular to the base ABC at point X, the tetrahedron is



Let $ABCD$ have unit sides. The centroid divides medians in the ratio 1 : 2, so $|AX| = \sqrt{3}/3$. Hence, triangle DAX is



By Pythagoras, $|DX| = \frac{\sqrt{6}}{3}$. So,

$$\theta = \arctan\frac{\sqrt{6}/3}{\sqrt{3}/3} = \sqrt{2}, \text{ as required.}$$

4350. Let the population have distribution

$$X \sim N(\mu, \sigma^2).$$

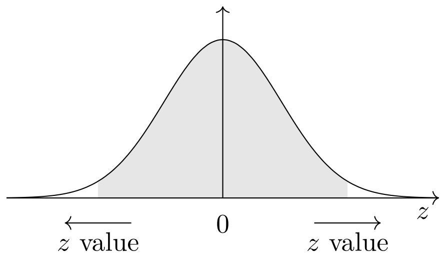
The distribution of the sample means is then

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{N}\right).$$

The z value associated with the probability in question is therefore

$$\begin{aligned} z &= \frac{a}{\frac{\sqrt{\sigma}}{N}} \\ &= \frac{aN}{\sqrt{\sigma}}. \end{aligned}$$

The values of a and σ are fixed. Hence, irrespective of their values, as $N \to \infty$, $z \to \infty$.



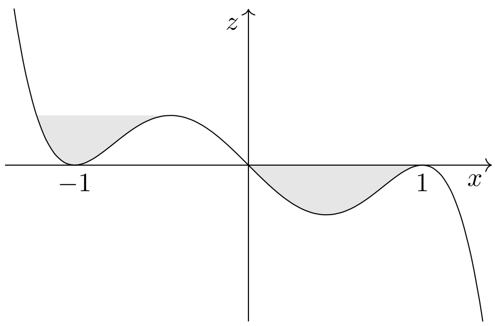
As N tends to ∞, the likelihood tends to certainty. Algebraically, this is

$$\lim_{N\to\infty} \mathbb{P}\left(|\bar{X} - \mu| < a\right) = 1.$$

4351. (a) Factorising, the profile is

$$z = -x(x-1)^2(x+1)^2.$$

So, the profile has double roots at $x = \pm 1$ and a single root at the (x, z) origin. Adding axes, we have



The cross-sectional area of the lower pool is

$$\begin{aligned} A &= -\int_0^1 -x^5 + 2x^3 - x\,dx \\ &= \left[\tfrac{1}{6}x^6 - \tfrac{1}{2}x^4 - 2x\right]_0^1 \\ &= \tfrac{1}{6}. \end{aligned}$$

The stream is 3 metres wide, so the volume of the lower pool is $3 \times \frac{1}{6} = 0.5$ m^3.

(b) For stationary points,

$$\begin{aligned} &-5x^4 + 6x^2 - 1 = 0 \\ \Longrightarrow\ & x = \pm 1, \pm\tfrac{1}{\sqrt{5}}. \end{aligned}$$

The relevant SP is $x = -1/\sqrt{5}$. At this point,

$$z = \tfrac{16}{25\sqrt{5}}.$$

So, this is the equation of the surface of the upper pool. To find the domain of the upper pool, we solve

$$-x^5 + 2x^3 - x - \tfrac{16}{25\sqrt{5}} = 0.$$

This quintic is not analytically solvable. So, we use the N-R method. The iteration is

$$x_{n+1} = x_n - \frac{-x_n^5 + 2x_n^3 - x_n - \frac{16}{25\sqrt{5}}}{-5x_n^4 + 6x_n^2 - 1}.$$

With $x_0 = -1.5$, we get $x_1 = -1.33941...$ and then $x_n \to -1.21847....$ So, the volume of the upper pool is given by

$$\begin{aligned} V &= 3\int_{-1.21847}^{-\frac{1}{\sqrt{5}}} \tfrac{16}{25\sqrt{5}} + x^5 - 2x^3 + x\,dx \\ &= 3 \times 0.11643... \\ &= 0.349\text{m}^3 \text{ (3sf)}. \end{aligned}$$

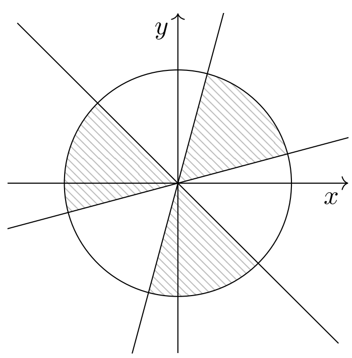
4352. The angles of inclination (measured anticlockwise from the positive x axis) of the lines are

Line	Angle
$y = -x$	$\arctan(-1) = -45°$
$y = (2 + \sqrt{3})$	$\arctan(2 + \sqrt{3}) = 75°$
$y = (2 - \sqrt{3})$	$\arctan(2 - \sqrt{3}) = 15°$

So, the lines pass through the origin, producing six radii at angles

$$15°, 75°, 135°, 195°, 255°, 315°.$$

These are in AP with common difference 60°, so they divide the circle into six sectors of equal area.



4353. (a) Yes: $y = x^{100}$ is below $y = x^{98} + 1$ at $x = 0$, and above it for large x.

(b) Yes: $y = x^{100}$ is below $y = x^{99} + 1$ at $x = 0$, and above it for large positive x.

(c) No: the curves are translations of one another in the y direction.

(d) Yes: $y = x^{100}$ is below $y = x^{101} + 1$ at $x = 0$, and above it for large negative x.

(e) No: $y = x^{100}$ is below $y = x^{102} + 1$ for all x.

4354. Let $x = \arcsin t$, so that $t = \sin x$. For $t \in [0, 1]$, $x \in [0, \pi/2]$. For these x values,

$$\cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - t^2}.$$

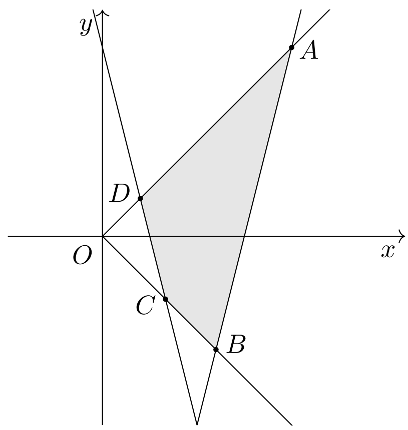
This gives

$$\tan x = \frac{\sin x}{\cos x} = \frac{t}{\sqrt{1 - t^2}}.$$

We can now integrate by inspection:

$$\begin{aligned} &\int_0^1 \tan(\arcsin t)\,dt \\ =& \int_0^1 \frac{t}{\sqrt{1 - t^2}}\,dt \\ =& \left[-\sqrt{1 - t^2}\right]_0^1 \\ =& (0) - (-1) \\ =& 1, \text{ as required.} \end{aligned}$$

4355. The graphs are



The vertices of the quadrilateral are

	A	B	C	D
x	5	3	$5/3$	1
y	5	-3	$-5/3$	1

$\triangle OAB$ is right-angled, with area

$$\tfrac{1}{2} \cdot 5\sqrt{2} \cdot 3\sqrt{2} = 15.$$

$\triangle OCD$ is also right-angled, with area

$$\tfrac{1}{2} \cdot \tfrac{5}{3}\sqrt{2} \cdot 1\sqrt{2} = \tfrac{5}{3}.$$

So, the area of $ABCD$ is $15 - \frac{5}{3} = \frac{40}{3}$, as required.

4356. (a) Both models give $r = 0$ at $t = 0$.

(b) For the models to coincide at $t = 10$,

$$\begin{aligned} 10^3 &= e^{10k} - 1 \\ \Longrightarrow e^{10k} &= 1001 \\ \Longrightarrow k &= \tfrac{1}{10}\ln 1001 \\ &= 0.6908... \\ &\approx \ln 2. \end{aligned}$$

(c) The total number of new infections is given by the integral of the rate with respect to t. Using the definite integration facility on a calculator, the values predicted are

① Cubic model:

$$\int_0^{10} t^3\, dt = 2500.$$

② Exponential model:

$$\int_0^{10} e^{t\ln 2} - 1\, dt \approx 1466.$$

The percentage difference, with the established cubic model as baseline, is

$$\begin{aligned} \frac{1466 - 2500}{2500} &= -41.36... \\ &\approx -41\%, \text{ as required.} \end{aligned}$$

— Nota Bene —

This type of problem, in which a rate is cumulated to produce a total, is why integral notation is what it is. The symbol is a big curly S, which stands for *sum*. It is the continuous version of the discrete Σ (the Greek S) which also stands for *sum*.

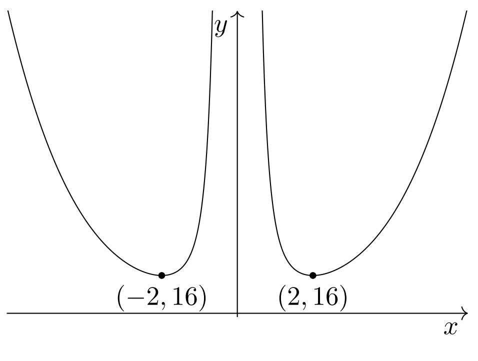
4357. (a) The numerator $x^2 + 4$ is positive everywhere. Hence, there are no x intercepts.

(b) Setting the derivative to zero for SPs,

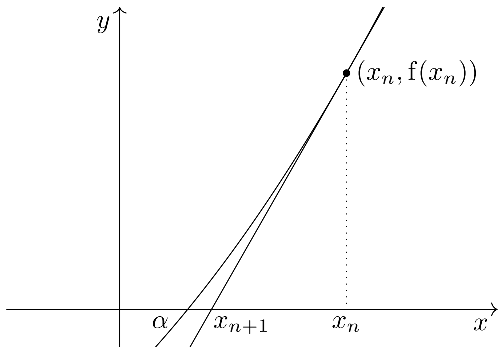
$$\begin{aligned} \frac{(2x)(2x) - 2(x^2+4)}{(2x)^2} &= 0 \\ \Longrightarrow 2x^2 - 8 &= 0 \\ \Longrightarrow x &= \pm 2. \end{aligned}$$

So, there are stationary points at $(\pm 2, 16)$.

(c) The curve has even symmetry and a quadruple asymptote at $x = 0$. It is positive everywhere.



4358. The Newton-Raphson iteration generates a new approximation x_{n+1} to a root α at the x intercept of the tangent line at x_n.



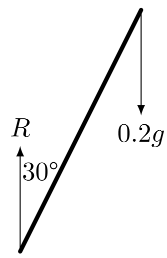
The gradient of the tangent line is $f'(x_n)$. Using the right-angled triangle formed beneath it,

$$f'(x_n) = \frac{\Delta y}{\Delta x} = \frac{f(x_n)}{x_n - x_{n+1}}.$$

Rearranging this,

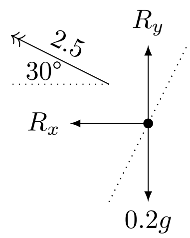
$$\begin{aligned} x_n - x_{n+1} &= \frac{f(x_n)}{f'(x_n)} \\ \Longrightarrow x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)}, \text{ as required.} \end{aligned}$$

4359. (a) The force diagram for the bar, not including the moment applied at the hinge, is



In this diagram, the clockwise moment around the hinge is $0.2g \sin 30° \times 0.1 = 0.098$ Nm. So, the moment applied by the hinge is 0.098 Nm clockwise, i.e. towards vertical.

(b) The force diagram for the bob is as follows. We resolve the contact force on the bob into components R_x and R_y. The acceleration of the bob is perpendicular to the bar, so it is inclined at 30° to the horizontal.



Resolving horizontally,

$$R_x = 0.2 \cdot 2.5 \cos 30° = \tfrac{\sqrt{3}}{4}.$$

Resolving vertically,

$$R_y - 0.2g = 0.2 \cdot 2.5 \sin 30° \implies R_y = 2.21.$$

So, the magnitude of the force exerted is

$$R = \sqrt{R_x^2 + R_y^2} = 2.25 \text{ N (3sf)}.$$

4360. Using a conditioning approach, place A_1 without loss of generality. With probability $\frac{2}{5}$, A_2 leaves a gap of one; with probability $\frac{1}{5}$, A_2 sits opposite. We address these case by case:

① $A_1{*}A_2{*}{*}{*}$. Someone has to sit in the single gap between A_1 and A_2. Call that person B_1. For success, B_2 must then, with probability $\frac{1}{3}$, sit in the middle of the gap, which makes $A_1B_1A_2{*}B_2{*}$. The C couple is then sorted.

② $A_1{*}{*}A_2{*}{*}$. We can place B_1 without loss of generality. There are now two successful positions for B_2, which have probability $\frac{2}{3}$. Again, the C couple is then sorted.

This gives $p = \frac{2}{5} \cdot \frac{1}{3} + \frac{1}{5} \cdot \frac{2}{3} = \frac{4}{15}$.

— Alternative Method —

Call the people $A_1, A_2, B_1, B_2, C_1, C_2$. There are $6! = 720$ seating plans. Of these, the successful outcomes are of two types:

Type	Example
All couples opposite	$A_1B_1C_1A_2B_2C_2$
One couple opposite	$A_1B_1C_1A_2C_2B_2$

We address these case by case:

① With all couples opposite, there are $6 \times 4 \times 2$ ways of filling the first three seats, after which the remaining seats are fixed. This gives 48 seating plans.

② With only the A couple opposite, there are 6 ways of placing A_1 and A_2. There are then 4 symmetrical ways of placing B_1. For each, B_2's position is fixed. There are then 2 ways of placing the C couple. This gives $6 \times 4 \times 2 = 48$ outcomes.

There are also symmetrical versions with only the B and only the C couple opposite. This gives $48 \times 3 = 144$ outcomes.

Putting it all together, $p = \frac{48+144}{720} = \frac{4}{15}$.

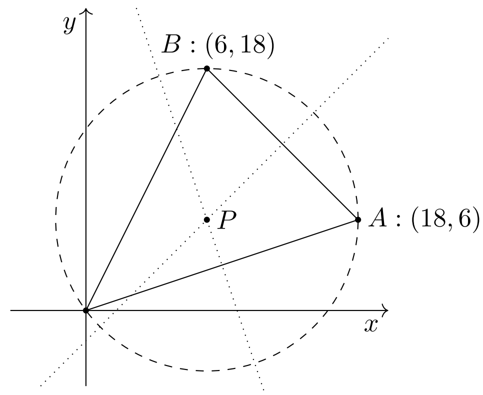
4361. Expanding with a compound-angle formula,

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0} \frac{\tan(x+h) - \tan(x)}{h} \\ &\equiv \lim_{h\to 0} \frac{\frac{\tan x + \tan h}{1 - \tan x \tan h} - \tan(x)}{h} \\ &\equiv \lim_{h\to 0} \frac{\tan x + \tan h - \tan(x)(1 - \tan x \tan h)}{h(1 - \tan x \tan h)} \\ &\equiv \lim_{h\to 0} \frac{\tan h(1 + \tan^2 x)}{h(1 - \tan x \tan h)}.\end{aligned}$$

Since h is small, $\tan h \approx h$. This gives

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0} \frac{h(1+\tan^2 x)}{h(1 - h\tan x)} \\ &\equiv \lim_{h\to 0} \frac{1 + \tan^2 x}{1 - h \tan x} \\ &\equiv 1 + \tan^2 x \\ &\equiv \sec^2 x, \text{ as required.}\end{aligned}$$

4362. To maximise the least distance to a vertex, we need point P to be equidistant from O, A, B. So, P is the circumcentre of the triangle:



The perpendicular bisector of AB is $y = x - 2$, and of OA is $y = -3x + 30$. Solving these for the intersection, point P is $(8, 6)$.

4363. There are $r_1 + r_2$ counters. So, the number of ways of choosing $r_1 + r_2$ squares on which to place them is

$$^n\mathrm{C}_{r_1+r_2} = \frac{n!}{(r_1+r_2)!(n - r_1 - r_2)!}.$$

Having placed them, there are r_1 squares to choose from $r_1 + r_2$. This is

$$^{r_1+r_2}\mathrm{C}_{r_1} = \frac{(r_1+r_2)!}{r_1!\,r_2!}.$$

Multiplying these gives

$$\frac{n!}{(r_1+r_2)!(n-r_1-r_2)!} \times \frac{(r_1+r_2)!}{r_1!\,r_2!} \equiv \frac{n!}{r_1!\,r_2!\,(n-r_1-r_2)!}.$$

— Alternative Method —

A choice of squares consists of a partition of the n squares into three groups: r_1 for black counters, r_2 for white counters, $(n-r_1-r_2)$ to be left blank. In the list of $n!$ orders of n squares, we overcount partitions by factors of $r_1!, r_2!$ and $(n-r_1-r_2)!$, corresponding to the orders within the partitioned groups. This gives

$$\frac{n!}{r_1!\,r_2!\,(n-r_1-r_2)!}.$$

4364. The first equation is a quadratic in xy:

$$\begin{aligned}&(3xy-1)(xy+1)=0\\ \Longrightarrow\ & xy=\tfrac{1}{3},-1.\end{aligned}$$

Solving $x+2y=1$ simultaneously with $xy={}^1\!/\!_3$ gives no real roots. With $xy=-1$ gives (x,y) points $(-1,1)$ and $(2,-{}^1\!/\!_2)$.

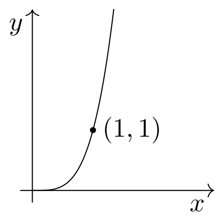
4365. This is true.

In a 2D square, the diagonal has length $\sqrt{2}$. In a 3D cube, the space diagonal has length $\sqrt{3}$. By Pythagoras, the pattern continues, giving $\sqrt{4}=2$ in a *tesseract* (4D hypercube) and then $\sqrt{5}$ in a *penteract* (5D hypercube).

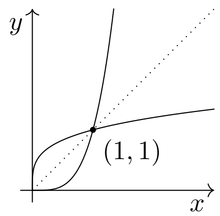
4366. Using the identity $\log_y x \equiv \frac{1}{\log_x y}$,

$$\begin{aligned}&\log_x y+\frac{1}{\log_x y}=4\\ \Longrightarrow\ &(\log_x y)^2-4\log_x y+1=0\\ \Longrightarrow\ &\log_x y=\tfrac{4\pm\sqrt{12}}{2}=2\pm\sqrt{3}\\ \Longrightarrow\ &y=x^{2\pm\sqrt{3}}.\end{aligned}$$

The curve is symmetrical in $y=x$; this is seen in the fact that $2-\sqrt{3}$ is the reciprocal of $2+\sqrt{3}$. In the positive quadrant, the graph of $y=x^{2+\sqrt{3}}$ is



The original relation has no points outside of the positive quadrant, as logarithms with negative bases/inputs are not defined. Reflecting the above graph in the line $y=x$, the full sketch is



— Nota Bene —

A real logarithm with a negative base **and also** a negative input is definable in some cases. The index statement $(-2)^3=-8$ translates to the log statement $\log_{-2}(-8)=3$, which has a well defined meaning. But, because $(-2)^2=4$, we can't define $\log_{-2}(-4)$ in the same manner.

Hence, because there is no *consistent* definition available, it is customary not to use real-valued logarithms with negative bases.

4367. (a) Every number in $(0,1)$ is reduced by squaring, so $p=1$.

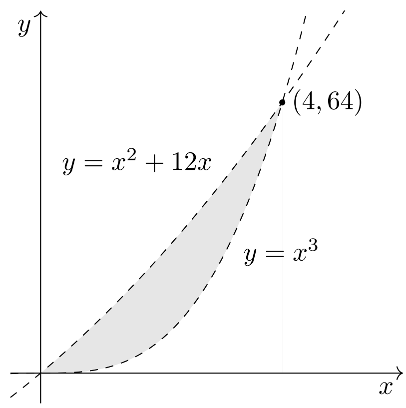
(b) Rearranging, the inequality is

$$\left(k-\tfrac{1}{2}\right)^2<0.$$

LHS ≥ 0, so there are no values of k which satisfy the inequality. This gives $p=0$.

(c) Solving, $\frac{1}{4}<k<\frac{3}{4}$. Since this is half of the possibility space, $p=\frac{1}{2}$.

4368. Sketching the boundary equations,



Setting up the relevant definite integral, the area of the region enclosed is

$$\int_0^4 x^2+12x-x^3\,dx=\tfrac{160}{3}.$$

4369. We know that $\mathrm{f}''(x)=\mathrm{g}''(x)$. Integrating this,

$$\mathrm{f}'(x)+a=\mathrm{g}'(x)+b.$$

Integrating again,

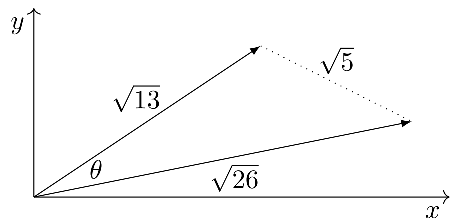
$$\begin{aligned}&\mathrm{f}(x)+ax+c=\mathrm{g}(x)+bx+d,\\ \Longrightarrow\ &\mathrm{f}(x)-\mathrm{g}(x)=(b-a)x+d-c.\end{aligned}$$

So, the equation $\mathrm{f}(x)=\mathrm{g}(x)$ can be rewritten as $\mathrm{f}(x)-\mathrm{g}(x)=0$, which is

$$(b-a)x+d-c.$$

Since this is a linear equation, it has, depending on the values of the constants a,b,c,d, either zero, one, or infinitely many roots. QED.

4370. Calculating the lengths of the vectors using Pythagoras, the scenario is



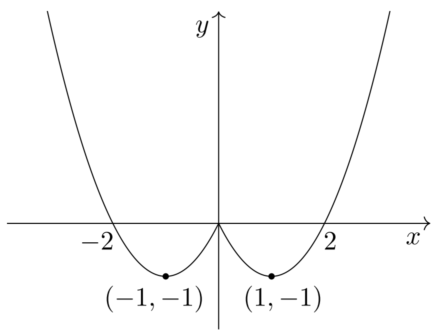
The cosine rule gives

$$\cos\theta = \frac{13+26-5}{2\sqrt{13}\sqrt{26}} = \frac{17\sqrt{2}}{26}.$$

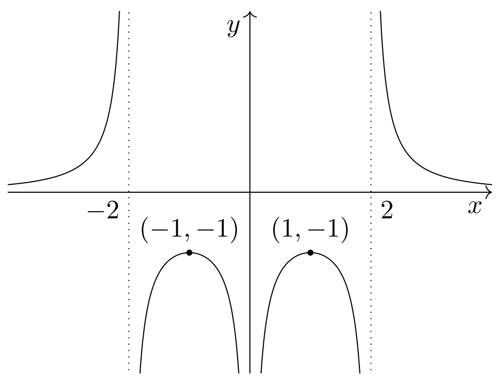
Using the definition of the dot product,

$$\begin{aligned}&\begin{pmatrix}3\\2\end{pmatrix}\cdot\begin{pmatrix}5\\1\end{pmatrix}\\ &= \left|\begin{pmatrix}3\\2\end{pmatrix}\right|\left|\begin{pmatrix}5\\1\end{pmatrix}\right|\cos\theta\\ &= \sqrt{13}\times\sqrt{26}\times\frac{17\sqrt{2}}{26}\\ &= 17, \text{ as required.}\end{aligned}$$

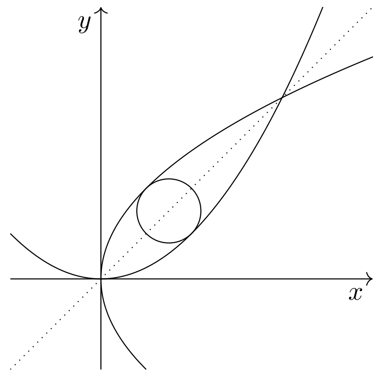
4371. Consider $y = x^2 - 2|x|$. Looking at the $x \geq 0$ and $x < 0$ cases separately, the graph is



Upon reciprocating the y values, we have single asymptotes at $x = \pm 2$, local maxima at $x = \pm 1$ and an asymptote at $x = 0$. As far as its rate of divergence is concerned, this is a single asymptote. However, $x^2 - 2|x|$ doesn't change sign at $x = 0$; so, in behaviour, the asymptote at $x = 0$ is akin to a double asymptote. Also, the x axis is a horizontal asymptote.



4372. The graphs are symmetrical in $y = x$. So, the centre of the circle must lie on this line.



The circle's diameter is maximised where the curves are parallel to $y = x$. Setting $\frac{dy}{dx} = \frac{1}{2}x = 1$, this is at $(2, 1)$ for $4y = x^2$.

By symmetry, the corresponding point is $(1, 2)$ for $4x = y^2$. So, the diameter is $\sqrt{2}$, as required.

4373. For $n \in \mathbb{Z}$, let the equation be

$$nx^2 + (n+1)x + (n+2) = 0.$$

The discriminant is

$$\begin{aligned}\Delta &= (n+1)^2 - 4n(n+2)\\ &\equiv 4 - 3(n+1)^2.\end{aligned}$$

For real roots, this must be non-negative:

$$\begin{aligned}&4 - 3(n+1)^2 \geq 0\\ \Longrightarrow\ &(n+1)^2 \leq \tfrac{4}{3}.\end{aligned}$$

The only squares which are less than $\frac{4}{3}$ are zero and one. Solving for n,

$$\begin{aligned}&(n+1)^2 = 0, 1\\ \Longrightarrow\ &n+1 = 0, \pm 1\\ \Longrightarrow\ &n = -2, -1, 0.\end{aligned}$$

But these values produce $c = 0$, $b = 0$ and $a = 0$ respectively, and we are told that $a, b, c \neq 0$. So, Q has no real roots, as required.

4374. (a) The initial vertical speed is $u\sin\theta$. So,

$$\begin{aligned}&0 = u^2\sin^2\theta - 2gh\\ \Longrightarrow\ &\sin^2\theta = \frac{2gh}{u^2}.\end{aligned}$$

(b) The time of flight is given by

$$\begin{aligned}&0 = (u\sin\theta)t - \tfrac{1}{2}gt^2\\ \Longrightarrow\ &t = 0 \text{ or } t = \frac{2u\sin\theta}{g}.\end{aligned}$$

Hence, the range is

$$\begin{aligned} d &= \frac{2u^2 \sin\theta\cos\theta}{g} \\ &= \frac{2u^2 \sin\theta\sqrt{1-\sin^2\theta}}{g} \\ &= \frac{2u^2 \frac{\sqrt{2gh}}{u}\sqrt{1-\frac{2gh}{u^2}}}{g} \\ &\equiv \frac{\sqrt{8gh(u^2-2gh)}}{g}, \text{ as required.} \end{aligned}$$

4375. There are ${}^4\mathrm{C}_3 = 4$ ways of selecting the three and and ${}^4\mathrm{C}_2 = 6$ ways of selecting the pair. So, the probability is

$$p = \frac{13 \times 12 \times {}^4\mathrm{C}_3 \times {}^4\mathrm{C}_2}{{}^{52}\mathrm{C}_5} = 0.144\% \text{ (3sf)}.$$

——— Alternative Method ———

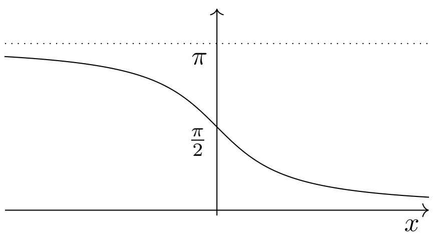
The cards must be AAABB in some order. The probability of choosing AAABB in that order is

$$1 \cdot \tfrac{3}{51} \cdot \tfrac{2}{50} \cdot \tfrac{48}{49} \cdot \tfrac{3}{48} = \tfrac{6}{41650}.$$

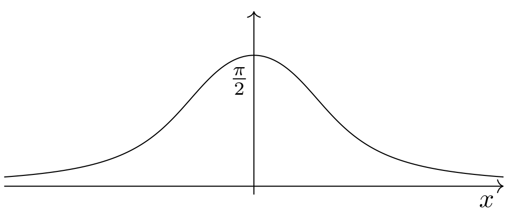
We multiply by the number of orders of AAABB, which is ${}^5\mathrm{C}_2 = 10$. This gives

$$p = 10 \times \tfrac{6}{41650} = 0.144\% \text{ (3sf)}.$$

4376. The graph of $y = \operatorname{arccot} x$ is



Replacing the input x by x^2 gives the graph even symmetry. Hence the y intercept at $(0,1)$ becomes a stationary point, and, as $x \to \pm\infty$, $y \to 0$:



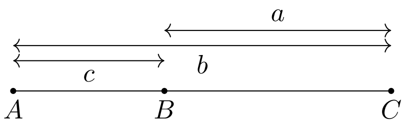
4377. (a) Using the cosine rule,

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

Since θ is small, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. This gives

$$\begin{aligned} c^2 &\approx a^2 + b^2 - 2ab\big(1 - \tfrac{1}{2}ab\theta^2\big) \\ &= a^2 + b^2 - 2ab + a^2b^2\theta^2 \\ &= (a-b)^2 + a^2b^2\theta^2. \end{aligned}$$

(b) As $\theta \to 0$, the right-hand term $a^2b^2\theta^2 \to 0$. So, in the limit, $c^2 \approx (a-b)^2$, which we can rewrite as $c \approx |a-b|$. We can interpret this as follows. In the limit, points A, B, C become collinear. Flattening to a number line:



In the limit of an infinitely thin triangle, the distances c and $|a-b|$ are the same.

4378. The expansions in the numerator are

$$(ax \pm h)^n = a^n x^n \pm na^{n-1}x^{n-1}h + \ldots.$$

The expansions in the denominator are

$$(bx \pm h)^n = b^n x^n \pm nb^{n-1}x^{n-1}h + \ldots.$$

So, the numerator is

$$2na^{n-1}x^{n-1}h + \ldots,$$

and the denominator is

$$2nb^{n-1}x^{n-1}h + \ldots.$$

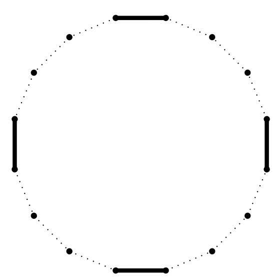
All remaining terms have factors of h^2. So,

$$\begin{aligned} L &= \lim_{h\to 0} \frac{2na^{n-1}x^{n-1}h + \ldots}{2nb^{n-1}x^{n-1}h + \ldots} \\ &\equiv \lim_{h\to 0} \frac{a^{n-1} + \ldots}{b^{n-1} + \ldots}. \end{aligned}$$

All remaining terms have factors of h. Hence, when we take the limit,

$$L = \frac{a^{n-1}}{b^{n-1}}, \text{ as required.}$$

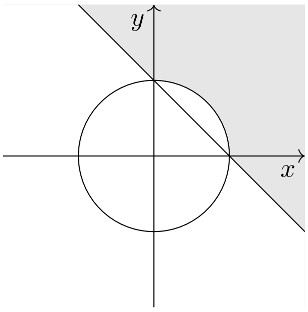
4379. For the edges to lie on the edges of a square, they must be rotationally symmetrical with order 4.



We choose the first edge without loss of generality. This fixes the possibilities for the remaining three, which gives

$$\begin{aligned} p &= 1 \times \frac{3}{n-1} \times \frac{2}{n-2} \times \frac{1}{n-3} \\ &\equiv \frac{6}{n^3 - 6n^2 + 11n - 6}, \text{ as required.} \end{aligned}$$

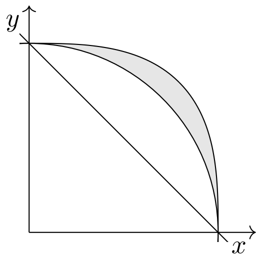
4380. The first two boundary equations are a straight line and a circle. So, the points satisfying the first two inequalities are



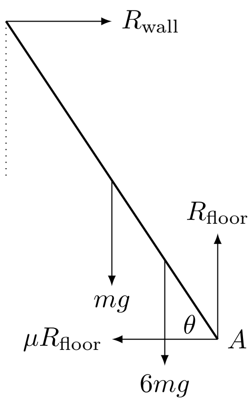
The third boundary equation, $x^3 + y^3 = 1$, can be sketched as follows:

- In the positive quadrant, the behaviour is akin to $x^2 + y^2 = 1$. All points lie on or outside the unit circle.
- There are no points in the negative quadrant.
- In the other two quadrants, the curve is asymptotic to $x + y = 0$: the addition of 1 become negligible for large x.
- The tangent is parallel to the x axis at the y intercept and parallel to the y axis at the x intercept.

Putting these together, the region in question is restricted to the first quadrant, lying between the curves $x^2 + y^2 = 1$ and $x^3 + y^3 = 1$:



4381. The force diagram, assuming limiting friction, is



Vertical equilibrium gives $\mathbb{R}_{\text{floor}} = 7mg$. Taking moments around A,

$$\begin{aligned} &R_{\text{wall}} l \sin\theta - \tfrac{1}{2} mgl\cos\theta - \tfrac{6}{4} mgl\cos\theta = 0 \\ \Longrightarrow\ &R_{\text{wall}} = 2mg\cot\theta. \end{aligned}$$

Using the right-angled triangle of which the ladder is the hypotenuse,

$$\cot\theta = \frac{d}{\sqrt{l^2 - d^2}}.$$

Horizontal equilibrium gives

$$\begin{aligned} &\mu \times 7mg = 2mg\cot\theta \\ \Longrightarrow\ &\mu = \frac{2d}{7\sqrt{l^2 - d^2}}. \end{aligned}$$

This is the value of μ for limiting friction. The ladder is in equilibrium, so we know that

$$\mu \geq \frac{2d}{7\sqrt{l^2 - d^2}}.$$

4382. The gradient of the normal at $x = p$ is $-\frac{1}{2p}$. So, the equation of the normal is

$$\begin{aligned} &y - p^2 = -\tfrac{1}{2p}(x - p) \\ \Longrightarrow\ &y = -\tfrac{1}{2p}x + p^2 + \tfrac{1}{2}. \end{aligned}$$

Solving for intersections with $y = x^2$,

$$\begin{aligned} &x^2 = -\tfrac{1}{2p}x + p^2 + \tfrac{1}{2} \\ \Longrightarrow\ &x^2 + \tfrac{1}{2p}x - p^2 - \tfrac{1}{2} = 0 \\ \Longrightarrow\ &2px^2 + x - 2p^3 - p = 0. \end{aligned}$$

This is a quadratic. The formula gives

$$\begin{aligned} x &= \frac{-1 \pm \sqrt{1 + 8p(2p^3 + p)}}{4p} \\ &\equiv \frac{-1 \pm (4p^2 + 1)}{4p} \\ &\equiv p, -p - \tfrac{1}{2p}. \end{aligned}$$

So, $q = -p - \frac{1}{2p}$. This gives

$$\begin{aligned} &q^2 - p^2 \\ &= \left(-p - \tfrac{1}{2p}\right)^2 - p^2 \\ &\equiv p^2 + 1 + \tfrac{1}{4p^2} - p^2 \\ &\equiv 1 + \tfrac{1}{4p^2}. \end{aligned}$$

The second term is non-zero, and is a square, so it is positive. Hence, $q^2 - p^2 > 1$, as required.

4383. Simplifying the indices,

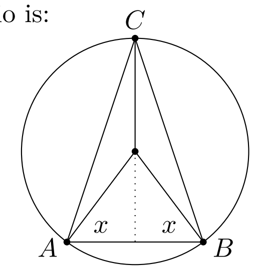
$$2a^{\frac{2}{3}} + a^{\frac{1}{3}} - 1 = 0.$$

This is a quadratic in $a^{\frac{1}{3}}$:

$$\begin{aligned} &(2a^{\frac{1}{3}} - 1)(a^{\frac{1}{3}} + 1) = 0 \\ \Longrightarrow\ &a^{\frac{1}{3}} = \tfrac{1}{2}, -1 \\ \Longrightarrow\ &a = \tfrac{1}{8}, -1. \end{aligned}$$

4384. (a) As in the diagram, we can take the base AB of the triangle to be horizontal. It is fixed in length. So, the area of the triangle depends only on its vertical height. By symmetry, the greatest height is on the perpendicular bisector of AB. This makes an isosceles triangle.

(b) The scenario is:



The dotted perpendicular shown has length $\sqrt{r^2 - x^2}$. So, $\triangle ABC$ has height $r + \sqrt{r^2 - x^2}$. This gives

$$\begin{aligned} A_\triangle &= \tfrac{1}{2}(2x)\left(r + \sqrt{r^2 - x^2}\right) \\ &\equiv x\left(r + \sqrt{r^2 - x^2}\right). \end{aligned}$$

(c) Optimising, we set $\frac{d}{dx}(A_\triangle) = 0$:

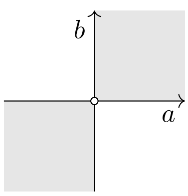
$$\begin{aligned} & r + \sqrt{r^2 - x^2} - \frac{x^2}{\sqrt{r^2 - x^2}} = 0 \\ \Longrightarrow\ & r\sqrt{r^2 - x^2} = 2x^2 - r^2 \\ \Longrightarrow\ & r^2\left(r^2 - x^2\right) = 4x^4 - 4r^2x^2 + r^4 \\ \Longrightarrow\ & 4x^4 - 3r^2x^2 = 0 \\ \Longrightarrow\ & x^2\left(4x^2 - 3r^2\right) = 0. \end{aligned}$$

We reject $x = 0$, which is a minimum of area. Taking the positive value, $x = \sqrt{3}/2$, which makes the internal angles $60°$. Hence, the area is maximised if $\triangle ABC$ is equilateral. □

4385. For f to be invertible over $\mathbb{R}$, it must be be one-to-one. So, we need there to be no sign change in the first derivative. The first derivative is $5ax^4 + 3bx^2$, which factorises as

$$\frac{dy}{dx} = x^2(5ax^2 + 3b).$$

The factor x^2 cannot produce a sign change, so we can ignore it. We require that $5ax^2 + 3b$ has no sign change, i.e. we require that the quadratic equation $5ax^2 + 3b = 0$ has a maximum of one root. This is the case if a and b are both non-negative or both non-positive.



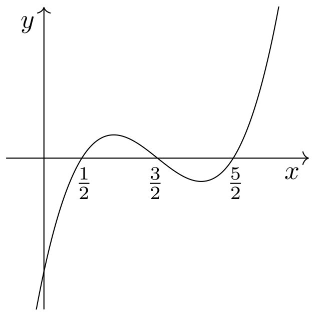
——— Nota Bene ———

The origin is excluded on the above graph, because $a = 0$ and $b = 0$ gives the zero function $f(x) = 0$. This is not invertible, as every input maps to zero.

4386. Since the roots are in arithmetic progression, i.e. symmetrically spaced, the central root must be the point of inflection of $y = 8x^3 - 36x^2 + 46x - k$. Setting the second derivative to zero, $48x - 72 = 0$ gives $x = \frac{3}{2}$. Substituting in and equating to zero,

$$\begin{aligned} & 8\left(\tfrac{3}{2}\right)^3 - 36\left(\tfrac{3}{2}\right)^2 + 46\left(\tfrac{3}{2}\right) - k = 0 \\ \Longrightarrow\ & k = 15. \end{aligned}$$

Solving, the roots are $x = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$.



4387. (a) With the integrating factor $I = \sec x$,

$$\begin{aligned} \tfrac{d}{dx}(\ln I) &= \tfrac{d}{dx}\left(\ln(\sec x)\right) \\ &\equiv \frac{\sec x \tan x}{\sec x} \\ &\equiv \tan x, \text{ as required.} \end{aligned}$$

(b) Multiplying by $\sec x$, the DE is

$$\frac{dy}{dx}\sec x + y \sec x \tan x = 0.$$

Since $(\sec x)' = \sec x \tan x$, we can write this as $(y \sec x)' = 0$, which is in the form required.

(c) Integrating, $y \sec x = A$, where A is constant. Multiplying by $\cos x$, the general solution is $y = A\cos x$, as required.

4388. There are ${}^{52}\text{C}_4 = 270725$ possible hands. Success requires one of two cases: three of one suit and one of another, or two pairs.

① There are 4 ways of choosing the suit of three, and then ${}^{13}\text{C}_3$ ways of choosing the cards. There are 39 ways of choosing the last card. This gives $4 \times {}^{13}\text{C}_3 \times 39 = 44616$ successful hands.

② There are ${}^{4}\text{C}_2 = 6$ ways of choosing the two suits. Once they are chosen, there are ${}^{13}\text{C}_2$ ways of choosing each pair. This gives $6 \times {}^{13}\text{C}_2 \times {}^{13}\text{C}_2 = 36504$ successful hands.

This gives the probability of success p as

$$p = \frac{44616 + 36504}{270725} = 0.300 \text{ (3sf)}.$$

——— Alternative Method ———

Using a conditioning approach, the successful cases are as above.

① The probability of AAAB in that order is

$$1 \times \tfrac{12}{51} \times \tfrac{11}{50} \times \tfrac{39}{49}.$$

There are 4 orders of AAAB, giving

$$p_1 = 4 \times \tfrac{12}{51} \times \tfrac{11}{50} \times \tfrac{39}{49} = 0.1648...$$

② The probability of AABB in that order is

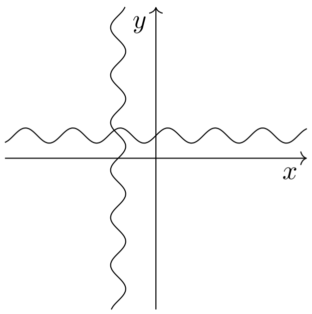
$$1 \times \tfrac{12}{51} \times \tfrac{39}{50} \times \tfrac{12}{49}.$$

Since suits A and B play symmetrical roles, there are only 3 orders of AABB, suit A being defined as "whatever is dealt first". This gives

$$p_2 = 3 \times \tfrac{12}{51} \times \tfrac{39}{50} \times \tfrac{12}{49} = 0.1348...$$

So, $p_1 + p_2 = 0.1648 + 0.1348 = 0.300$ (3sf).

4389. Consider the curves on a large scale: $y = \sin x + p$ is on average parallel to the x axis, and $x = \sin y + q$ is on average parallel to the y axis. So, they must cross somewhere:

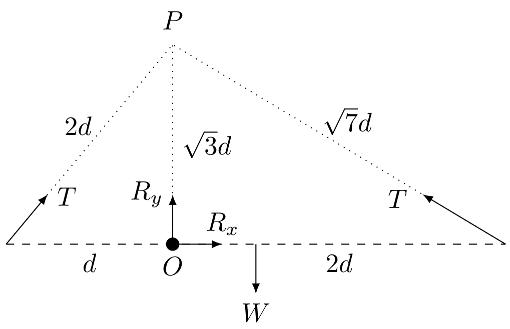


Consider the range of the gradient of each curve.

- The first curve has derivative $\frac{dy}{dx} = \cos x$, so its gradient takes values $[-1, 1]$.
- For the second curve, by symmetry, the range of $\frac{dx}{dy}$ is $[-1, 1]$, so its gradient takes values $(\infty, -1] \cup [1, \infty)$.

In other words, the second curve is always as steep or steeper than the first curve. Hence, they cannot intersect more than once. QED.

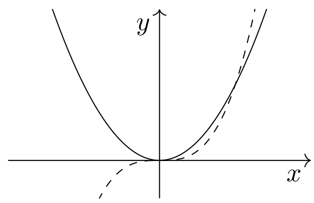
4390. Using Pythagoras to calculate various lengths, the force diagram for the bar is



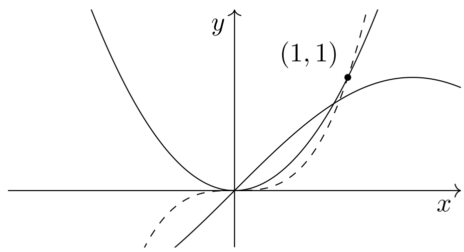
For moments about O, we resolve each tension into its horizontal and vertical components. Around O, horizontal components have no moment. Using the lengths in the diagram,

$$\begin{aligned}
& T\tfrac{\sqrt{3}}{2} \cdot d + W \cdot \tfrac{1}{2}d - T\tfrac{\sqrt{3}}{\sqrt{7}} \cdot 2d = 0 \\
\Longrightarrow\ & T\left(\tfrac{2\sqrt{3}}{\sqrt{7}} - \tfrac{\sqrt{3}}{2}\right) = \tfrac{1}{2}W \\
\Longrightarrow\ & T = \frac{\frac{1}{2}}{\left(\frac{2\sqrt{3}}{\sqrt{7}} - \frac{\sqrt{3}}{2}\right)}W \\
& \equiv \frac{\frac{1}{2}\left(\frac{2\sqrt{3}}{\sqrt{7}} + \frac{\sqrt{3}}{2}\right)}{\frac{12}{7} - \frac{3}{4}}W \\
& \equiv \frac{7\sqrt{3} + 4\sqrt{21}}{27}W, \text{ as required.}
\end{aligned}$$

4391. We need the inequalities $x^2 < \sin x$ and $\sin x < x^3$ to be satisfied simultaneously. This needs $x^2 < x^3$. The graphs of $y = x^2$ and $y = x^3$ are



So, $x^2 < x^3$ for $x \in (1, \infty)$. Over this domain, the range of x^2 is $(1, \infty)$ and the range of $\sin x$ is $[-1, 1]$. Hence, there are no x values in this domain for which $x^2 < \sin x$. So, there are no x values that satisfy $x^2 < \sin x < x^3$.

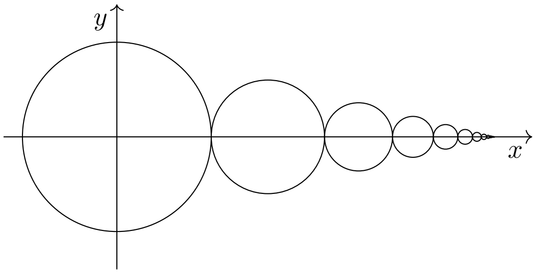


4392. The input to the mod function $(x+1)$ changes sign at $x = -1$. This is not in the integral domain. So, the mod function is passive throughout, and can be ignored. We integrate by parts. Let $u = x + 1$ and $v' = e^{-x}$, so that $u' = 1$ and $v = -e^{-x}$. Using the parts formula,

$$\begin{aligned}
& \int_0^\infty |x+1|\, e^{-x}\, dx \\
&= \Big[-(x+1)e^{-x}\Big]_0^\infty + \int_0^\infty e^{-x}\, dx \\
&= \Big[-(x+1)e^{-x} - e^{-x}\Big]_0^\infty
\end{aligned}$$

The exponential decay is more powerful than the linear factor, so evaluating at the upper limit ∞ gives 0. Evaluating at the lower limit 0 gives -2, so the integral has value 2, as required.

4393. Since the total length is finite, the GP must be decreasing. No circle lies inside another, so the circles are tangent to one another in an increasing sense, i.e. one after another along the x axis. The scenario, not to scale, is as follows:



Let the nth diameter be d_n. This is a GP with first term d and common ratio r. We know two facts about these diameters:

- The sum of all the diameters is 100. This is the infinite sum of a GP. So,

$$100 = \frac{d}{1-r}.$$

- The combined area of the first two circles is $\frac{181}{4}\pi$. The first two radii are $\frac{1}{2}d$ and $\frac{1}{2}dr$, so

$$\begin{aligned} \tfrac{181}{4}\pi &= \pi\left(\tfrac{1}{2}d\right)^2 + \pi\left(\tfrac{1}{2}dr\right)^2 \\ \implies 181 &= d^2\left(1+r^2\right). \end{aligned}$$

Solving these equations simultaneously, we get two solutions. One has a negative diameter, which we discard. The other is $d = 10$, $r = 0.9$. Hence, the diameter of the second circle is 9.

4394. (a) If the particle is to stay on a single parabola as shown, then that parabola must intersect $y = \sqrt{3}/3|x|$ at right angles. The line has gradient $\pm\sqrt{3}/3$, so the parabola must have gradient $\mp\sqrt{3}$ at these points.

(b) The surfaces have $y = 0$ as a line of symmetry, so the parabola must do too. Hence, the speed of 0.7 ms^{-1} must be horizontal. In the positive quadrant, travelling rightwards, the motion is $x = 0.7t$, $y = h - \frac{1}{2}gt^2$. The equation of the trajectory is

$$y = h - \tfrac{1}{2}g\left(\tfrac{x}{0.7}\right)^2 = h - 10x^2.$$

So, $\frac{dy}{dx} = -20x$. Setting this to $-\sqrt{3}$,

$$-\sqrt{3} = -20x \implies x = \tfrac{\sqrt{3}}{20}.$$

At this point, the y coordinate is $\frac{\sqrt{3}}{3}\cdot\frac{\sqrt{3}}{20} = \frac{1}{20}$. Substituting into the parabola,

$$\begin{aligned} \tfrac{1}{20} &= h - 10\left(\tfrac{\sqrt{3}}{20}\right)^2 \\ \implies h &= \tfrac{1}{20} + \tfrac{3}{40} \\ &= \tfrac{1}{10}. \end{aligned}$$

So, when $x = 0$, $y = 0.1$ m.

4395. (a) Multiplying by d and b, the equations are

$$\begin{aligned} adx + bdy &= dp \\ bcx + bdy &= bq. \end{aligned}$$

Subtracting these, we get $(ad - bc)x = dp - bq$. If $ad - bc \neq 0$, then we can divide by it, giving

$$x = \frac{dp - bq}{ad - bc}.$$

A similar argument solving for y produces the same condition: $ad - bc \neq 0$. We can rewrite this as

$$\frac{a}{c} \neq \frac{b}{d}.$$

(b) If $\frac{a}{c} = \frac{b}{d}$, i.e. $ad = bc$, then the equations are

$$\begin{aligned} adx + bdy &= dp \\ adx + bdy &= bq. \end{aligned}$$

For infinitely many (x, y) solution points, the equations must be the same, with $dp = bq$. Rearranging this, the condition is

$$\frac{a}{c} = \frac{b}{d} = \frac{p}{q}.$$

4396. The quantity $2x+y$ is maximised in the direction of the line $y = \frac{1}{2}x$. Solving this simultaneously with $x^2 + y^2 = 1$, we get $x = \pm 2/\sqrt{5}$ and $y = \pm 1/\sqrt{5}$. The value of $2x + y$ is $\sqrt{5}$.

4397. Using a compound-angle formula,

$$\begin{aligned} \sin\left(\tfrac{\pi}{4} \pm \tfrac{x}{2}\right) &\equiv \sin\tfrac{\pi}{4}\cos\tfrac{x}{2} \pm \cos\tfrac{\pi}{4}\sin\tfrac{x}{2} \\ &\equiv \tfrac{\sqrt{2}}{2}(\cos\tfrac{x}{2} \pm \sin\tfrac{x}{2}). \end{aligned}$$

So, the LHS of the proposed identity is

$$\begin{aligned} &\sin\left(\tfrac{\pi}{4} + \tfrac{x}{2}\right)\operatorname{cosec}\left(\tfrac{\pi}{4} - \tfrac{x}{2}\right) \\ \equiv\ &\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}. \end{aligned}$$

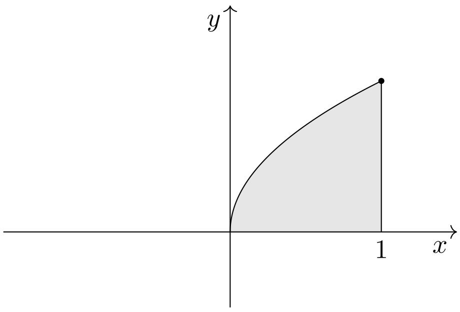
Multiplying top and bottom by the conjugate of the denominator, this is

$$\begin{aligned} &\frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}} \\ \equiv\ &\frac{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}} \\ \equiv\ &\frac{1 + 2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}. \end{aligned}$$

Using two double-angle formulae, this is

$$\begin{aligned} &\frac{1 + \sin 2\left(\frac{1}{2}x\right)}{\cos 2\left(\frac{1}{2}x\right)} \\ \equiv\ &\frac{1 + \sin x}{\cos x} \\ \equiv\ &\sec x + \tan x, \text{ as required.} \end{aligned}$$

4398. The Cartesian equation of the curve is $x = y^2$. So, the relevant region is



The area is $\int_0^1 y^2 \, dy = \frac{1}{3}$.

4399. The (restricted) possibility space consists of all strictly ascending triples. There are ${}^6\text{C}_3 = 20$ sets of three numbers from $\{1, 2, 3, 4, 5, 6\}$, and each has exactly one strictly ascending order. So, the possibility space is 20 equally likely outcomes.

With 6 as the last digit, the successful outcomes are the ${}^5\text{C}_2 = 10$ strictly ascending pairs from $\{1, 2, 3, 4, 5\}$.

So, the probability is $\frac{10}{20} = \frac{1}{2}$.

4400. (a) The position is given by

$$x = \int -t^2 e^{-t} \, dt.$$

We use the tabular integration method:

Signs	Derivatives	Integrals
$+$	$-t^2$	e^{-t}
$-$	$-2t$	$-e^{-t}$
$+$	-2	e^{-t}
$-$	0	$-e^{-t}$

This gives

$$\begin{aligned} x &= t^2 e^{-t} + 2te^{-t} + 2e^{-t} + c \\ &\equiv (t^2 + 2t + 2)e^{-t} + c. \end{aligned}$$

Substituting $t = 0$ and $x = 2$, we get $c = 0$. So, the position is $x = (t^2 + 2t + 2)e^{-t}$.

(b) The exponential factor is always positive. So, to reach zero, we would require $t^2 + 2t + 2 = 0$. But this has discriminant $\Delta = -4 < 0$. So, although in the long term the particle does tend asymptotically to position $x = 0$, it never reaches $x = 0$, as required.

——— End of 44th Hundred ———